

# Higher Dimensional Geometries from Matrix Brane Constructions

Pei-Ming Ho<sup>1</sup>, Sanjaye Ramgoolam<sup>2</sup>

<sup>1</sup> *Department of Physics  
National Taiwan University  
Taipei 106, Taiwan, R.O.C.*

<sup>2</sup> *Department of Physics  
Brown University  
Providence, RI 02912*

`pmho@phys.ntu.edu.tw`, `ramgosk@het.brown.edu`

Matrix descriptions of even dimensional fuzzy spherical branes  $S^{2k}$  in Matrix Theory and other contexts in Type II superstring theory reveal, in the large  $N$  limit, higher dimensional geometries  $SO(2k+1)/U(k)$ , which have an interesting spectrum of  $SO(2k+1)$  harmonics and can be up to 20 dimensional, while the spheres are restricted to be of dimension less than 10. In the case  $k=2$ , the matrix description has two dual field theory formulations. One involves a field theory living on the non-commutative coset  $SO(5)/U(2)$  which is a fuzzy  $S^2$  fibre bundle over a fuzzy  $S^4$ . In the other, there is a  $U(n)$  gauge theory on a fuzzy  $S^4$  with  $\mathcal{O}(n^3)$  instantons. The two descriptions can be related by exploiting the usual relation between the fuzzy two-sphere and  $U(n)$  Lie algebra. We discuss the analogous phenomena in the higher dimensional cases, developing a relation between fuzzy  $SO(2k)/U(k)$  cosets and unitary Lie algebras.

*Nov 2001*

## 1. Introduction

Fuzzy spheres of dimension higher than two have found a variety of physical applications [1][2][3][4][5] and may be relevant to the ADS/CFT correspondence [6][7][8][9]. In this paper we will study some geometrical aspects of the matrix algebras which are related to these higher fuzzy spheres. This will allow us to observe some interesting features associated with the physics of fluctuations around solutions of brane actions involving these fuzzy spheres.

In section two we review some background on higher dimensional fuzzy spheres. Fuzzy spheres are described by looking for an  $N \times N$  matrix realization of the equation

$$\sum_{\mu=1}^{2k+1} X_{\mu} X_{\mu} = 1,$$

which classically describes a sphere. The  $X_{\mu}$  matrices transform in the vector representation of  $SO(2k+1)$  symmetry group of the sphere. The decomposition of the matrix algebras into irreducible representations of  $SO(2k+1)$  is reviewed. It has been pointed out that only a subset of these representations approaches the space of spherical harmonics on a classical sphere [10]. This raises the issue of the geometry of the large  $N$  limit of the full matrix algebra. Some relevant mathematical results in harmonic analysis are noted, which indicate that the relevant space is  $SO(2k+1)/U(k)$ .

In section three we describe some aspects of the system of algebraic equations obeyed by the generators of the matrix algebra. The  $X_{\mu}$ 's are among the generators and they are obtained from the action of gamma matrices  $\Gamma_{\mu}$  of  $SO(2k+1)$  on  $Sym(V^{\otimes n})$ , an irreducible representation of  $SO(2k+1)$  obtained by symmetrizing the  $n$ -fold tensor power of the fundamental spinor. This representation is  $N$  dimensional where  $N \sim n^{k(k+1)}$ , and the  $X^{\mu}$  are  $N \times N$  matrices. The matrix algebra is denoted as  $\hat{\mathcal{A}}_n(S^{2k})$ . While the  $X_{\mu}$  do generate  $\hat{\mathcal{A}}_n(S^{2k})$  under matrix multiplication at finite  $n$ , consideration of the large  $n$  limit indicates that it is useful to introduce extra generators transforming in antisymmetric tensor representations of  $SO(2k+1)$ , and obeying some constraint equations. We describe further geometrical structures on  $\hat{\mathcal{A}}_n(S^{2k})$ , namely an algebra of derivations  $SO(2k+1, 1)$ . It is useful to distinguish the matrix algebra  $\hat{\mathcal{A}}_n(S^{2k})$  from a subspace  $\mathcal{A}_n(S^{2k})$  which decomposes into symmetric traceless representations of  $SO(2k+1)$ . This subspace  $\mathcal{A}_n(S^{2k})$  has a non-associative multiplication which becomes associative in the large  $N$  limit [10]. It is shown that  $SO(2k+1)$  acts as derivations of this non-associative algebra. Further properties of  $\mathcal{A}_n(S^{2k})$  are developed in the Appendix.

In section four we consider, in detail, the fuzzy four-sphere. We show that the large  $n$  limit of  $\hat{\mathcal{A}}_n(S^4)$  describes the algebra of functions on  $SO(5)/U(2)$ . This is done by considering a special solution to the system of equations obeyed by the generators of the matrix algebra in the large  $N$  limit. Rather than writing out all the equations and looking for solutions, we use the simple fact that the explicit matrix construction of the generators should give solutions to the equations. Following the relation between operators and classical variables in quantum mechanics, we obtain these solutions by considering the expectation values of these generators in a state. Given the solution, we find its stabilizer group to be  $U(2)$ . Since the system of equations obeyed by the generators is  $SO(5)$  covariant, the special solution generates  $SO(5)/U(2)$ . The harmonics on this space coincide with the large  $N$  decomposition of the matrix algebra, thus showing that the matrix algebra  $\hat{\mathcal{A}}_n(S^4)$ , approaches the space of functions on this coset.

The coset  $SO(5)/U(2)$  is a bundle over  $S^4$  with fibre which is a two-sphere. We describe a finite  $n$  version of this bundle structure, with the fibre being a fuzzy two-sphere. The connection between the fuzzy two-sphere and  $U(n+1)$  is exploited to show that the antisymmetric generators  $X_{ij}$  which generate the algebra of functions on the fibre can be associated with field strengths of a  $U(n+1)$  bundle at a point on the  $S^4$ . In this way, we establish a direct relation, by construction so to speak, between the matrix description of the worldvolume theory of  $N \sim n^3/6$   $D0$  branes with the field strengths of the dual instanton description in terms of  $U(n+1)$  gauge theory with  $N \sim n^3/6$  instantons on the worldvolume of  $n+1$  spherical 4-branes, a connection expected on general physical grounds [2][4].

In section five and six we develop analogous points for the fuzzy  $S^6$  and the fuzzy  $S^8$ . This uses an extension of the connection between unitary Lie algebras and fuzzy two-spheres [11] to connections between unitary Lie algebras and  $SO(2k)/U(k)$  for  $k = 3, 4$ . This is expected to generalize to any  $k$ .

In section seven, we observe that fluctuations of branes described in the fuzzy sphere constructions have in addition to a description in terms of non-Abelian gauge theory on spheres with a large number of instantons, a description in terms of a field theory on the  $SO(2k+1)/U(k)$  coset. The field theory has symmetries which include a non-commutative  $U(1)$  gauge symmetry and a global  $SO(2k+1)$  symmetry. A complete understanding of the higher dimensional action is left for the future.

In the Appendix we show that the non-associative algebra  $\mathcal{A}_n(S^{2k})$  is actually commutative at finite  $n$ . If all  $\mathcal{O}(1/n)$  terms are dropped from the structure constants, the

algebra is commutative and associative and coincides with the algebra of spherical harmonics. If  $\mathcal{O}(1/n)$  terms are kept, and  $\mathcal{O}(1/n^2)$  terms are dropped, then the algebra is an infinite dimensional Jordan Algebra. At higher orders in  $1/n$  and at finite  $n$ , it is a more general commutative and non-associative algebra which is not Jordan. For concreteness these points are developed for the case  $k = 2$  but the arguments are of a general nature.

## 2. Review of relevant background

Here we will review some material from physics and mathematics literature which will be useful.

### 2.1. Fuzzy spheres and matrix brane constructions

The fuzzy four-sphere was used in the construction of time-dependent 4-brane solutions from the action of zero-branes in [2], in the context of Matrix Theory of BFSS. On the other hand, the Lagrangian for a matrix model coupled to totally antisymmetric tensor field of rank  $(2k + 1)$  is of the form [12]

$$L = Tr \left( \frac{1}{2} [\Phi_\mu, \Phi_\nu]^2 + F \epsilon_{\mu_1 \dots \mu_{2k+1}} \Phi_{\mu_1} \dots \Phi_{\mu_{2k+1}} \right),$$

where  $F$  is the field strength. An example of this was found by Myers [12] that D0-branes expand into spherical branes in the background of RR gauge fields. Along similar lines, one considers the worldvolume action for D-strings ending on D5-branes and in this case 5 matrices transverse to the D-string worldvolume describe the end-point acquiring a spherical 4-dimensional geometry[4]. A different application involves the study of compactifications of M-theory in the context of matrix model compactified on a  $(2k + 1)$ -dimensional sphere. The Lagrangian is of the form

$$L = Tr \left( \frac{1}{2} [\Phi_\mu, \Phi_\nu]^2 - a (\Phi_\mu \Phi_\mu - R^2) \right),$$

where  $a$  is the Lagrange multiplier. The  $X_\mu$  which are used to construct fuzzy sphere coordinates in the next section can be used to solve the equations of motion in all of the above cases by setting  $\Phi_\mu = LX_\mu$  for some constant  $L$ .

Since the algebra of functions of  $X$  is equivalent to the algebra of  $N \times N$  matrices, in these cases, generic configurations of the matrix variables can be described as fields living on a manifold whose algebra of functions is approximated by the large  $N$  limit of  $\hat{\mathcal{A}}_n(S^{2k})$ . In this paper, we will describe this manifold, and relate its finite  $n$  non-commutative geometry to non-Abelian gauge theory on the fuzzy sphere. The rank and instanton number of the gauge bundle are both related to  $N$  which sets the scale of non-commutativity, so these constructions are far from simply being discretized versions of spheres.

## 2.2. Decomposing matrix algebras into reps. of $SO(2k+1)$

The construction of fuzzy spheres  $S^{2k}$  starts with the action of  $X_\mu$ ,  $\mu = 1 \cdots 2k+1$ , which are obtained from an action of  $\Gamma_\mu$  on  $Sym(V^{\otimes n})$ :

$$X_\mu = \sum_{r=1}^n \rho_r(\Gamma_\mu). \quad (2.1)$$

The operator  $\rho_r(\Gamma_\mu)$  acts on the  $r$ -th copy of  $V$  in  $V^{\otimes n}$ . Since the sum (2.1) is symmetric it gives a well defined action on  $Sym(V^{\otimes n})$ . Each  $X_\mu$  is an  $N \times N$  matrix where  $N \sim n^{k(k+1)}$  at large  $n$ . Products of the  $X_\mu$  generate the full set of matrices. Different products can be used to generate sets of matrices transforming according to different representations of  $SO(2k+1)$ . Each tensor representation occurs once, with a cutoff on the first row of the Young diagram corresponding to the representation of  $r_1 \leq n$ . With this cutoff, the dimensions of these representations add up to  $N^2$ , exactly the size of the matrix algebra [10]. In the large  $n$  limit, all representations of  $SO(2k+1)$  occur with unit multiplicity.

The algebra of functions on the sphere  $S^{2k}$  decomposes into representations which are symmetric and traceless, i.e they correspond to Young diagrams with row lengths  $r_i = 0$  for  $i > 1$ . To get this algebra we have to project out the representations which have any columns of length greater than 1. On this projected space we have a non-associative multiplication.

In this paper we will be focusing on the geometry of the full matrix algebra. Since the matrix algebra has a simple decomposition into representations of  $SO(2k+1)$ , this can be expected to give some strong hints about the geometry related to it.

## 2.3. A heuristic guess

The operators  $X_\mu$  have eigenvalues ranging from  $-n$  to  $n$ . The number of degrees of freedom in the matrix algebra grows like  $N^2$  which is  $n^{k(k+1)}$ . If we think of the space described by the full matrix algebra as a discretized space and associate one degree of freedom with each point, we expect the space to be  $k(k+1)$  dimensional. This will indeed be the dimension of the coset space  $SO(2k+1)/U(k)$  which we will prove to have an algebra of functions which is the classical limit of the matrix algebra.

## 2.4. The results of Kramer

A paper of Kramer [13] has done the job of finding the space whose algebra of functions contains each representation of  $SO(2k+1)$  once. It is the space  $SO(2k+1)/U(k)$ . This space has dimension  $k(k+1)$ . The construction of algebras which contain each representation of  $SO(2k+1)$  is not in fact unique. Such constructions are referred to as model spaces, and they can also come up as spaces of sections of certain bundles over cosets [14]. To establish that our matrix algebra is indeed associated with  $SO(2k+1)/U(k)$  we will describe the system of equations obeyed by these matrices in the large  $n$  limit.

## 3. Description in terms of algebraic equations

### 3.1. Explicit form of gamma matrices

To fix notation we will give the form of the gamma matrices  $\Gamma_\mu$  with  $\mu = 1 \cdots 2k+1$ . The gamma matrices of  $SO(2k+1)$  obey the equations :

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}. \quad (3.1)$$

They can be expressed in terms of a set of fermionic oscillators  $a_i$  with  $i$  running from 1 to  $k$ , and obeying

$$\{a_i, a_j^\dagger\} = \delta_{ij}. \quad (3.2)$$

The expressions for the gamma matrices are :

$$\begin{aligned} \Gamma_{2i-1} &= (a_i + a_i^\dagger), \\ \Gamma_{2i} &= i(a_i - a_i^\dagger) \end{aligned} \quad (3.3)$$

for  $i = 1, \dots, k$  and  $\Gamma_{2k+1} = \Gamma_1 \cdots \Gamma_{2k}$ .

A  $2^k$  dimensional representation of the  $\Gamma$  matrix algebra is obtained by defining a state  $|0\rangle$  which is annihilated by the fermionic annihilation operators  $a_i$  and acting with the creation operators to generate  $2^k$  different states.

### 3.2. Algebra of the fuzzy sphere

From the matrices describing the fuzzy spheres, we can write down a set of algebraic equations which are satisfied by the variables. In the large  $n$  limit all the variables involved are commutative.

For a fuzzy sphere define  $X_\mu$  as in (2.1). Let us denote their commutators by

$$X_{\mu\nu} = \frac{1}{2}[X_\mu, X_\nu] = \frac{1}{2} \sum_r \rho_r (\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu). \quad (3.4)$$

For the fuzzy 6-sphere we also need the totally antisymmetrized cubic products of gamma matrices: <sup>1</sup>

$$\begin{aligned} X_{\mu\nu\lambda} &= \sum_r \rho_r (\Gamma_{[\mu} \Gamma_\nu \Gamma_{\lambda]}) \\ &= \sum_r \rho_r (\Gamma_\mu \Gamma_\nu \Gamma_\lambda + \Gamma_\lambda \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\lambda \Gamma_\mu - \Gamma_\nu \Gamma_\mu \Gamma_\lambda - \Gamma_\lambda \Gamma_\nu \Gamma_\mu - \Gamma_\mu \Gamma_\lambda \Gamma_\nu) \end{aligned} \quad (3.5)$$

and similarly, for the fuzzy-8-sphere, we will also need

$$X_{\mu\nu\lambda\rho} = \sum_r \rho_r (\Gamma_{[\mu} \Gamma_\nu \Gamma_\lambda \Gamma_{\rho]}). \quad (3.6)$$

These matrices satisfy the following algebraic relations:

$$X_\mu X_\mu = c, \quad (3.7)$$

$$[X_{\mu\nu}, X_\rho] = 2(\delta_{\nu\rho} X_\mu - \delta_{\mu\rho} X_\nu), \quad (3.8)$$

$$[X_{\mu\nu}, X_{\kappa\rho}] = 2(\delta_{\nu\kappa} X_{\mu\rho} + \delta_{\mu\rho} X_{\nu\kappa} - \delta_{\mu\kappa} X_{\nu\rho} - \delta_{\nu\rho} X_{\mu\kappa}), \quad (3.9)$$

where

$$c = n(n + 2k). \quad (3.10)$$

The  $X_{\mu\nu}$ 's can be identified with generators of the  $SO(2k + 1)$  Lie algebra. Together with  $X_\mu$ , they generate the Lie algebra of  $SO(2k + 1, 1)$ .

In addition, we have generalized self-duality relations

$$\epsilon_{\mu_1 \dots \mu_{2k+1}} X_{\mu_1} \dots X_{\mu_{2k}} = \lambda X_{\mu_{2k+1}}. \quad (3.11)$$

---

<sup>1</sup> They are not antisymmetric cubic products of  $X_\mu$ 's, but are more complicated functions of them.

This relation is valid only when they are viewed as operators on  $Sym(V^{\otimes n})$ . For  $k = 1$ ,  $\lambda = 2$ . For  $k = 2$ ,  $\lambda = 8(n + 2)$ . For  $k = 3$ ,  $\lambda = 144n(n^2 + 4)$ .

To describe the geometry corresponding to this algebra, we define variables  $Z_\mu$ ,  $Z_{\mu\nu}$ ,  $Z_{\mu\nu\lambda}$ ,  $Z_{\mu\nu\lambda\rho}$ , which satisfy algebraic equations with real coefficients and they generate an algebra which defines a non-commutative manifold. They are related to  $X$  through the following relations

$$\begin{aligned} Z_\mu &\sim \frac{1}{n} X_\mu, \\ Z_{\mu\nu} &\sim \frac{i}{n} X_{\mu\nu}, \\ Z_{\mu\nu\lambda} &\sim \frac{1}{n} X_{\mu\nu\lambda}, \\ Z_{\mu\nu\lambda\rho} &\sim \frac{i}{n} X_{\mu\nu\lambda\rho}. \end{aligned} \tag{3.12}$$

The  $Z$ 's are suitably normalized so that they obey equations having coefficients  $\mathcal{O}(1)$  in the large  $n$  limit. In this paper, we will use  $Z$  as generators of the algebra of functions  $\hat{\mathcal{A}}_n(S^4)$  and use  $X$ 's as derivatives.

### 3.3. Derivations on the matrix algebra $\hat{\mathcal{A}}_n(S^{2k})$

In the large  $n$  limit the matrix algebra reduces to an algebra of polynomials in  $Z$ 's with constraints, which will be given later. The action of the operators  $X_\mu$ ,  $X_{\mu\nu}$  by commutators transforms the polynomials in the variables  $Z_\mu, Z_{\mu\nu}$ . In particular, the  $X_{\mu\nu}$ 's satisfy an  $SO(2k + 1)$  algebra, organizing all polynomials of  $Z$ 's into representations of  $SO(2k + 1)$ .

Recall that we chose the normalizations (3.12). We know from (3.4), (3.8) that

$$\begin{aligned} [X_\mu, Z_\nu] &= -2i Z_{\mu\nu}, \\ [X_\mu, Z_{\nu\lambda}] &= -2i(\delta_{\mu\nu} Z_\lambda - \delta_{\mu\lambda} Z_\nu). \end{aligned} \tag{3.13}$$

To reproduce these formulae we write :

$$ad(X_\mu) = -2i \left( Z_{\mu\nu} \frac{\partial}{\partial Z_\nu} + Z_\nu \frac{\partial}{\partial Z_{\mu\nu}} \right). \tag{3.14}$$

Similarly, we know from (3.8), (3.9) that

$$\begin{aligned} [X_{\mu\nu}, Z_\lambda] &= 2(\delta_{\nu\lambda} Z_\mu - \delta_{\mu\lambda} Z_\nu), \\ [X_{\mu\nu}, Z_{\lambda\rho}] &= 2(\delta_{\nu\lambda} Z_{\mu\rho} - \delta_{\mu\lambda} Z_{\nu\rho} + \delta_{\mu\rho} Z_{\nu\lambda} - \delta_{\nu\rho} Z_{\mu\lambda}). \end{aligned} \tag{3.15}$$



This can be reproduced by

$$ad(X_{\mu\nu}) = 2 \left( Z_\mu \frac{\partial}{\partial Z_\nu} - Z_\nu \frac{\partial}{\partial Z_\mu} - Z_{\mu\lambda} \frac{\partial}{\partial Z_{\lambda\nu}} + Z_{\nu\lambda} \frac{\partial}{\partial Z_{\lambda\mu}} \right). \quad (3.16)$$

For consistency, the equations (3.14) and (3.16) should agree with  $[X_\mu, X_\nu] = 2X_{\mu\nu}$ , and indeed they do.

The  $X_\mu, X_{\mu\nu}$  can also be realized as differential operators on the fuzzy sphere  $\mathcal{A}_n(S^{2k})$ . This is to be expected since we know that there is a dual description in terms of  $U(n)$  gauge theory on  $S^4$  with instantons. In the original matrix model,  $Z_\mu$  can be thought of as the matrix coordinates of D0-branes.

### 3.4. Derivations on $\mathcal{A}_n(S^{2k})$

We showed above that the action by commutators of  $X_\mu$  and  $X_{\mu\nu}$ , denoted as  $ad(X_\mu)$  and  $ad(X_{\mu\nu})$ , can be realized as derivatives on the finite  $n$  coset  $\hat{\mathcal{A}}_n(S^{2k})$ . We show here that  $ad(X_{\mu\nu})$  also acts as derivations on the non-associative algebra  $\mathcal{A}_n(S^{2k})$ , which is the finite  $n$  algebra of spherical harmonics. The product on this space is defined by first multiplying as matrices and then projecting onto symmetric representations. Indeed letting  $A$  and  $B$  be matrices in  $\mathcal{A}_n(S^{2k})$ , the product  $A \circ B$  is defined by

$$A \circ B = P(AB), \quad (3.17)$$

where  $P$  acting on any matrix transforming in anything other than a symmetric representation is zero, and leaves invariant any matrix transforming in the symmetric traceless representation [10]. From its definition it is clear that  $P$  commutes with the action of  $SO(2k+1)$  on matrices. Let us denote  $ad(X_{\mu\nu}) = L_{\mu\nu}$ . We can write

$$\begin{aligned} L_{\mu\nu}(A \circ B) &= L_{\mu\nu}(P(AB)) \\ &= P(L_{\mu\nu}(AB)) = P(L_{\mu\nu}(A)B + AL_{\mu\nu}(B)) \\ &= P(L_{\mu\nu}(A)B) + P(AL_{\mu\nu}(B)) \\ &= L_{\mu\nu}(A) \circ B + A \circ L_{\mu\nu}(B). \end{aligned} \quad (3.18)$$

This shows that  $L_{\mu\nu}$  acts as a derivation on  $\mathcal{A}_n(S^{2k})$ . This is a finite  $n$  generalization of the action of  $SO(2k+1)$  on functions on the sphere  $S^{2k}$ .

To obtain an action of the full set of generators entering the algebra (3.8), (3.9) we can also work with just the sphere, without invoking the coset, but we need to equip the

sphere with an embedding in  $\mathbf{R}^5$  and consider vector fields on the normal bundle. We will describe the classical ( large  $n$  ) version of this construction here.

In terms of the Cartesian coordinates  $x_\mu$  ( $\mu = 1, \dots, 2k+1$ ) in  $\mathbf{R}^{2k+1}$ , functions restricted to a unit sphere are functions of  $z_\mu \equiv x_\mu/|x|$ . The naive derivative  $\frac{\partial}{\partial z_\mu}$  is not well defined because it does not respect the constraint  $z_\mu z_\mu = 1$ .

Define vector fields on the normal bundle by

$$\partial_\mu = |x| \frac{\partial}{\partial x_\mu}.$$

We have

$$[\partial_\mu, z_\nu] = \delta_{\mu\nu} - z_\mu z_\nu = P_{\mu\nu}, \quad (3.19)$$

where  $P_{\mu\nu}$  is the projection operator which projects a vector in  $\mathbf{R}^{2k+1}$  at  $z$  to the tangent plane on  $S^{2k}$ . One can check that the constraint  $z_\mu z_\mu = 1$  is preserved under differentiation by  $\partial_\mu$ . It turns out that

$$[\partial_\mu, \partial_\nu] = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} = L_{\mu\nu}, \quad (3.20)$$

which is the generator of  $SO(2k+1)$  rotations. We see that  $(\partial_\mu, L_{\mu\nu})$  satisfy the same relations (3.8) and (3.9) as  $(X_\mu, X_{\mu\nu})$  (up to an overall factor of 2). We will leave a finite generalization of this construction to the future.

#### 4. The fuzzy 4-sphere case

Rather than analysing the complete set of equations and looking for explicit solutions, a more efficient method is to use the known matrices, and obtain classical quantities, by taking expectation values between states. One can check afterwards that the algebraic equations are indeed satisfied, as they should.

Take the state  $|s\rangle \equiv |0\rangle \otimes |0\rangle \cdots |0\rangle$  in  $Sym(V^{\otimes n})$  which is the  $n$ -fold tensor product of the fermion Fock space vacuum. Since  $\Gamma_5|0\rangle = -|0\rangle$ , we have  $X_5|s\rangle = -n|s\rangle$ . The normalized operator  $Z_5$  has an expectation value  $\langle s|Z_5|s\rangle = -1$ . We also find that  $Z_{12} = 1, Z_{34} = 1$ . So one solution to the system of algebraic equations that will be given in sec.4.4 is

$$\begin{aligned} Z_5 &= -1, \\ Z_{12} &= 1, \\ Z_{34} &= 1, \end{aligned} \quad (4.1)$$

with all other  $Z_\mu$  and  $Z_{\mu\nu}$  being zero. In the above we have abbreviated  $\langle s|Z|s\rangle$  as  $Z$ .

Now we look at the subgroup of  $SO(5)$  which leaves this solution fixed. Requiring that the configuration  $X_5 = -1$ ,  $X_i = 0$  for  $i = 1, 2, 3, 4$ , be left invariant leads to an  $SO(4)$  subgroup that acts non-trivially on the index  $i$ . Consider linear combinations  $\lambda = \sum_{ij} \lambda_{ij} L_{ij}$ , where the lambda matrix is displayed below :

$$\lambda = \begin{pmatrix} 0 & \lambda_{13} & \lambda_{12} & \lambda_{14} \\ -\lambda_{13} & 0 & \lambda_{14} & \lambda_{34} \\ -\lambda_{12} & -\lambda_{14} & 0 & \lambda_{13} \\ -\lambda_{14} & -\lambda_{34} & -\lambda_{13} & 0 \end{pmatrix}. \quad (4.2)$$

We have chosen for convenience to label the rows and columns in the order (1324). These matrices  $\sum_{ij} \lambda_{ij} L_{ij}$  take the form

$$A + B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \quad (4.3)$$

Here  $A$  is anti-symmetric and  $B$  is symmetric. These  $4 \times 4$  matrices form a  $U(2)$  subgroup of the  $SO(4)$  subgroup of  $SO(5)$  which leaves  $X_5$  invariant.

The generators  $L_{ij}$ , which are antisymmetric matrices of the  $SO(4)$  Lie algebra, act on  $Z_{kl}$  as follows :

$$L_{ij}(Z_{kl}) = \delta_{jk} Z_{il} + \delta_{il} Z_{jk} - \delta_{ik} Z_{jl} - \delta_{jl} Z_{ik}. \quad (4.4)$$

Consider a linear combination  $\zeta = \sum_{i,j} \zeta_{ij} Z_{ij}$ . Let a linear combination  $\sum_{ij} \lambda_{ij} L_{ij}$  act on  $\zeta$ . The result of this action is a sum  $\delta\zeta = \sum_{ij} V_{ij} Z_{ij}$ . The coefficients  $V$  depend bilinearly on  $\lambda$  and  $\zeta$ . Considering  $V_{ij}$  as the entries of a  $4 \times 4$  matrix  $V$ , (4.4) is equivalent to  $V = [\lambda, \zeta]$ . Decomposing  $\zeta$  into  $2 \times 2$  blocks as

$$\zeta = \begin{pmatrix} X & Y \\ -Y & \tilde{X} \end{pmatrix}, \quad (4.5)$$

we have  $X^T = -X$ ,  $\tilde{X}^T = -\tilde{X}$  while  $Y$  is arbitrary. For  $\lambda$  in the  $U(2)$  subalgebra (4.3) and using (4.5) ,

$$V = \begin{pmatrix} ([A, X] - [B, Y]) & ([A, Y] + B\tilde{X} - XB) \\ -BX + \tilde{X}B - [A, Y] & -[B, Y] + [A, \tilde{X}] \end{pmatrix}. \quad (4.6)$$

Now we want to evaluate the variation  $V$  on the solution we started with. The solution can be described by a matrix  $\zeta^{(s)}$  which is of the form

$$\zeta^{(s)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4.7)$$

where each entry is a  $2 \times 2$  block, and 0 is the zero matrix, and 1 is the identity matrix. Given the form of the solution (4.7), this is equal to the trace of the off-diagonal blocks of  $V$ , i.e  $V_{12} + V_{34}$ . Now  $\text{tr}([A, Y]) = 0$ . Further,  $B\tilde{X}$  and  $BX$  are each antisymmetric, given that  $B$  is symmetric and  $X, \tilde{X}$  are antisymmetric. Therefore, the off-diagonal block of the variation  $V$ , when  $\lambda$  is in the  $u(2)$  subalgebra, has zero trace. This means that the solution is invariant under the  $U(2)$  subgroup.

We have therefore shown that the stabilizer group of the solution is  $U(2)$ . The action of  $SO(5)$  then generates a space of solutions which is  $SO(5)/U(2)$ . The space related to the large  $N$  limit of the matrix algebra is then  $SO(5)/U(2)$ . This is a bundle over  $S^4$  with fibre  $SO(4)/U(2)$ , a symmetric space. Using the isomorphism  $SO(4) = (SU(2) \times SU(2))/Z_2$  it can be shown that the fibre is actually  $SU(2)/U(1)$ , i.e the two-sphere. So we have in this case

$$\begin{aligned} SO(5)/U(2) &\longleftarrow SO(4)/U(2) \equiv SU(2)/U(1) \equiv S^2 \\ &\downarrow \\ S^4 &\equiv SO(5)/SO(4) \end{aligned} \tag{4.8}$$

We can also work directly with (4.4) and show that the  $U(2)$  subgroup described in (4.2) and (4.3) leaves (4.1) invariant. The generators of  $U(2)$  are  $L_{12}, L_{34}, L_{13} + L_{24}$ , and  $L_{14} - L_{23}$ . The generators  $L_{13} - L_{24}, L_{12} + L_{34}$  and  $L_{14} + L_{23}$  generate the  $S^2$  bundle at  $|s\rangle$  which we will explain in more detail in the next subsection.

Another viewpoint toward this space is to view  $X_\mu$  and  $X_{\mu\nu}$  as generators of the  $SO(5, 1)$  Lie algebra. Similar arguments to the above lead to the conclusion that the space of the matrix algebra is the quotient space  $SO(5, 1)/U(2, 1) \simeq SO(5)/U(2)$ .

#### 4.1. Bundle structure at finite $n$

In the previous subsection we evaluated the operators  $Z_\mu$  and  $Z_{\mu\nu}$  for a state  $|s\rangle$  in  $\text{Sym}(V^{\otimes n})$ . By analysing the space of states which leaves the expectation values of  $Z_\mu$  invariant, we will identify the space of states corresponding to the fibre of the non-commutative  $SO(5)/U(2)$  coset over a fixed point of the sphere.

We first define the base space of the bundle in the following way. As pointed out in [10], only totally symmetrized polynomials of  $Z_\mu$  should be viewed as the quantum version of functions on  $S^4$ . Let us use the term *non-associative*  $S^4$  for the underlying space of this (non-associative) algebra of functions. This is our base space. The non-associative algebra of functions on the  $S^4$  will be referred to as  $\mathcal{A}_n(S^4)$ .

We define two states in the Hilbert space  $Sym(V^{\otimes n})$  of the matrix algebra to be on the same fiber if all functions in  $\mathcal{A}_n(S^4)$  have the same expectation values for both states. According to the definition of Connes [15], this implies that the distance <sup>2</sup> between the projection of the two states on  $\mathcal{A}_n(S^4)$  is zero.

The state  $|s\rangle$ , for example, is on the same fiber with all (normalized) states in the space  $W \equiv Sym(V_-^{\otimes n})$  where the space  $V_-$  is spanned by  $|0\rangle$  and  $a_1^\dagger a_2^\dagger |0\rangle$ . One can check that  $\langle s'|Z_\mu|s'\rangle$  is the same for all  $|s'\rangle \in Sym(V_-^{\otimes n})$ . For  $f$  in  $\mathcal{A}_n(S^4)$ ,  $\langle s'|f(Z)|s'\rangle$  is nonvanishing only if it is a product of an even number of  $Z_i$ 's ( $i = 1, \dots, 4$ ). In fact, the nonzero contributions are from terms with even numbers of  $\Gamma_i$ 's in each tensorial factor. However, a total symmetrization will render such products of  $\Gamma_i$ 's to be proportional to the unity. Since  $\langle v|\Gamma_5|v\rangle = -1$  for all normalized  $|v\rangle$  in  $V_-$ . We see that all functions on  $\mathcal{A}_n(S^4)$  have the same expectation values for all  $|s'\rangle$ .

The states on the fiber  $W$  form a representation for the operators  $M_1 = Z_{13} - Z_{24}$ ,  $M_2 = Z_{23} + Z_{14}$ ,  $M_3 = Z_{12} + Z_{34}$ , which satisfy the algebra of a fuzzy  $S^2$

$$[M_i, M_j] = -2i\epsilon_{ijk}M_k. \quad (4.10)$$

The rest of the  $Z_{\mu\nu}$ 's have vanishing expectation values on the fiber. We can thus view the fiber as a fuzzy 2-sphere with Cartesian coordinates  $M_i$ . Since the matrix algebra is covariant under  $SO(5)$  transformations, the same can be said about every point of the base space  $\mathcal{A}_n(S^4)$ . We conclude that the fuzzy 4-sphere is in fact a fuzzy  $S^2$  fiber bundle over the base space of  $\mathcal{A}_n(S^4)$ .

The points on the base sphere can be identified with the quotient space  $Sym(V^{\otimes n})/W$ . The projection map from  $Sym(V^{\otimes n})$  to the quotient space is the finite  $n$  analog of the projection map from the coset space  $SO(5)/U(2)$  to the sphere  $S^4$ .

Functions of  $M_i$  are sufficient to distinguish all states on the fiber. In fact,  $W$  is a  $(n+1)$  dimensional irreducible representation for the  $SU(2)$  Lie algebra generated by  $M_i$ . The set of all functions of  $M_i$  is the same as the set of all  $(n+1) \times (n+1)$  matrices.

---

<sup>2</sup> The distance between two states  $\psi, \psi'$  on a space with algebra of functions  $\mathcal{A}$  is defined to be

$$dist(\psi, \psi') = \sup\{|\langle \psi|f|\psi\rangle - \langle \psi'|f|\psi'\rangle| : |Df| \leq 1, f \in \mathcal{A}\}, \quad (4.9)$$

where  $D$  is the so-called Dirac operator which defines the differential calculus. Although we did not specify the Dirac operator for  $\mathcal{A}_n(S^4)$ , it should be clear that our comment here is valid for any choice of  $D$ .

Hermitian functions of  $M_i$  generate the Lie algebra of  $U(n+1)$ . This leads to a direct connection between the space of horizontal one-forms ( with indices parallel to the base ) on the  $SO(5)/U(2)$  coset with connections of a  $U(n+1)$  bundle over the sphere. Further, scalar fields on the coset correspond to adjoint scalars which are sections of the  $U(n+1)$  bundle.

#### 4.2. Relation to unitary bundle over $S^4$

Interpreting  $X_{\mu\nu} \propto Z_{\mu\nu}$  as a field strength, as is familiar in Matrix Theory, the fuzzy  $S^2$  fiber bundle can also be understood as a  $U(n+1)$  vector bundle on spherical D4-branes, whose worldvolume is the non-associative  $S^4$ .

This connection can be used, in the large  $n$  limit, to compare the  $X_{\mu\nu}$  matrix at the point  $Z_\mu = (0, 0, 0, 0, 1)$  with the field strengths of the homogeneous instanton. We have  $X_{12} = X_{34}$  while other field strengths are zero. We will use the fact that

$$\begin{aligned} 1 &= \langle 0 | \Gamma_1 \Gamma_2 | 0 \rangle = \langle 0 | \Gamma_3 \Gamma_4 | 0 \rangle, \\ -1 &= \langle 0 | a_2 a_1 \Gamma_1 \Gamma_2 a_1^\dagger a_2^\dagger | 0 \rangle = \langle 0 | a_2 a_1 \Gamma_3 \Gamma_4 a_1^\dagger a_2^\dagger | 0 \rangle, \end{aligned} \tag{4.11}$$

A basis in  $Sym(V_-^{\otimes n})$  can be labelled by the number of  $|0\rangle$ 's appearing in the tensor product. In such a basis it is clear that  $X_{12} = X_{34}$  is proportional to

$$Diag(n, n-2, \dots, -n+2, -n). \tag{4.12}$$

Compare this with the solution described in Appendix B of [4] where the  $F_{12} = F_{34}$  are the only non-zero field strengths at the point  $\alpha_1 = \frac{\pi}{2}$  with all other angles zero. They are proportional to the embedding of  $2\sigma_3$  in  $U(n+1)$  using the standard  $(n+1)$  dimensional representation of  $SU(2)$ , which is again the matrix (4.12) .

A physical interpretation of this space is the following. The  $Z_\mu$ 's can be viewed as matrix coordinates of  $N \simeq n^3/6$  D0-branes. We know that the solution given by matrices (3.13) also contains spherical 4-branes. The number of these 4-branes at large  $n$  can be computed using the definition of 4-brane charge in Matrix Theory [16] as done in [2]. In the application considered by [4] the zero-branes are replaced by 1 branes and the 4-branes by 5-branes. The number of 5-branes can be computed, in a manner similar to the Matrix Theory application, by analyzing couplings in the D1-brane action. Note that, in the previous paragraph, the number of D4 ( or D5 ) was extracted directly from the fuzzy fibre bundle structure of the  $\hat{\mathcal{A}}_n(S^{2k})$  without appealing to couplings on brane actions,

a fact which should be useful in more general contexts in Matrix Theory and D-brane physics.

In the dual description, on the worldvolume of  $n$  coincident spherical D4-branes, we have a  $U(n)$  gauge theory,  $X_\mu$ 's are interpreted as the  $U(n)$  covariant derivatives and  $X_{\mu\nu}$  as field strengths. D0-branes are realized as instantons of the  $U(n)$  gauge field. In (4.1) we see that the field strength  $X_{\mu\nu}$  is indeed a self-dual configuration at the north pole. Since the matrix algebra is invariant under  $SO(5)$ , it is clear that the matrices  $X_{\mu\nu}$  correspond to homogeneous instantons on  $S^4$ . It was found in [18][4] that the maximal number of homogeneous instantons for an  $U(n)$  bundle on  $S^4$  is  $n(n^2 - 1)/6$ . In the large  $n$  limit, this is precisely the instanton number for our configuration  $X$ , as first observed in [4].

Fluctuations around this configuration can be described in terms of fields on the coset  $SO(5)/U(2)$ . If a transverse matrix coordinate  $\Phi$  is excited to have a dependence on  $Z_{\mu\nu}$ , such as  $\Phi \sim Z_{\mu\nu}$ , a D2-brane dipole with charge density  $tr(Z_\mu Z_\nu \Phi)$  is generated [4]. This is because each D4-brane has a different D2-brane charge density. They add up to zero when the D4-branes are on top of each other. But if different D4-branes have different transverse fluctuations, D2-brane dipoles will appear.

#### 4.3. Traces and Integration

Integration on this non-commutative coset  $SO(5)/U(2)$  can be defined as the trace over  $Sym(V^{\otimes n})$ . The cyclicity of the trace automatically ensures  $SO(5)$  invariance of the integration because

$$Tr(L_{\mu\nu}(f(Z))) = \frac{1}{2}Tr([X_{\mu\nu}, f(Z)]) = 0.$$

For a given function of  $Z$ , one can decompose it into irreducible representations of  $SO(5)$  as [10]

$$f(Z) = \sum_{R,s} a_{R,s} Y_{R,s}(Z),$$

where  $R$  stands for irreducible representations and  $s$  for the indices of states in  $R$ . The  $SO(5)$  invariance implies that

$$\int d\Omega f(Z) = a_0 = \frac{1}{N} Tr(f),$$

where  $a_0$  is the coefficient for the trivial representation, which is unique (with multiplicity one). Hence, in the large  $n$  limit, this integration agrees with the usual integration on  $SO(5)/U(2)$  induced from the unique Haar measure on  $SO(5)$ . The measure  $d\Omega$  is normalized such that the volume of the coset is 1.

#### 4.4. Further remarks on the coset $SO(5)/U(2)$

Here we provide another way to obtain the same result of the coset  $SO(5)/U(2) \simeq S^4 \times S^2$  by directly investigating the matrix algebra. Some of the algebraic relations for the large  $n$  matrix algebra are

$$Z_\mu Z_\mu \simeq \mathbf{1}, \quad (4.13)$$

$$Z_{\mu\nu} Z_{\mu\nu} \simeq 4 \times \mathbf{1}, \quad (4.14)$$

$$\epsilon_{\mu\nu\lambda\rho\kappa} Z_{\mu\nu} Z_{\lambda\rho} \simeq -8Z_\kappa, \quad (4.15)$$

$$Z_{\mu\nu} Z_{\nu\lambda} \simeq -2\delta_{\mu\lambda} + Z_\mu Z_\lambda, \quad (4.16)$$

$$Z_\mu Z_{\mu\lambda} \simeq 0. \quad (4.17)$$

One can also show that

$$\epsilon_{\mu\nu\lambda\rho\kappa} Z_\mu Z_{\nu\lambda} \simeq 2Z_{\rho\kappa}, \quad (4.18)$$

which says that  $Z_{\mu\nu}$  is a self-adjoint tensor field on the sphere.

Relation (4.13) means that  $Z_\mu$  can be viewed as Cartesian coordinates of a unit fuzzy sphere. The identity (4.14) implies that the magnitude  $|Z_{\mu\nu}|$  is of order 1. Incidentally, since

$$[Z_\mu, Z_\nu] = \frac{-2i}{n} Z_{\mu\nu}, \quad (4.19)$$

we can estimate the uncertainty in  $Z_\mu$  as

$$\Delta Z_\mu \Delta Z_\nu \sim \mathcal{O}(1/n). \quad (4.20)$$

This means that “points” on the the unit fuzzy sphere with Cartesian coordinates  $Z_\mu$  have spread which scales like  $1/n$ .

In the large  $n$  limit, the commutator of  $Z_\mu$  (4.19) vanishes, showing that  $Z_{\mu\nu}$  has to be included as generators of the matrix algebra. All functions of  $Z_\mu$ ’s can be obtained from totally symmetrized products of  $Z_\mu$ ’s and  $Z_{\mu\nu}$ ’s. Yet the  $Z_{\mu\nu}$ ’s are not completely independent, but are constrained by (4.14), (4.15) and (4.18). Up to terms of order  $\mathcal{O}(1/n^2)$ , we can interpret these relations as constraints on a classical space with commutative coordinates  $Z_\mu, Z_{\mu\nu}$ . At a given point on  $S^4$ , say,  $(Z_1, \dots, Z_5) = (0, 0, 0, 0, 1)$ , the space of solutions to these constraints is the set of anti-self dual tensors  $Z_{ij}$  for  $i, j = 1, 2, 3, 4$ . This space is a 2-sphere. The whole space is thus locally  $S^4 \times S^2$ .



## 5. The fuzzy 6-sphere case

### 5.1. Solution to Equations and Stabilizer groups

Analogous to (4.1), the expectation values of  $|s\rangle$  are

$$\begin{aligned} Z_7 &= -1, \text{ and } Z_i = 0 \text{ for } i = 1 \cdots 6, \\ Z_{12} &= Z_{34} = Z_{56} = \frac{1}{\sqrt{6}} \text{ and } Z_{\mu\nu} = 0 \text{ otherwise,} \\ Z_{127} &= Z_{347} = Z_{567} = \frac{1}{\sqrt{18}} \text{ and } Z_{\mu\nu\rho} = 0 \text{ otherwise.} \end{aligned} \quad (5.1)$$

We have here defined  $Z$ 's so that they satisfy the normalization condition

$$\sum_{\mu} Z_{\mu}^2 = 1, \quad \sum_{\mu \neq \nu} Z_{\mu\nu}^2 = 1, \quad \sum_{\mu \neq \nu \neq \rho} Z_{\mu\nu\rho}^2 = 1. \quad (5.2)$$

The  $Z_{\mu\nu\rho}$ 's are antisymmetric in the three indices. Eq.(5.1) is a classical solution to (5.2) and self-adjoint relations analogous to (4.15) and (4.18).

We prove that the stabilizer group of this solution is  $U(3)$ . We define  $\zeta = \sum_{ij} \zeta_{ij} Z_{ij}$ . On the classical solution this takes the value

$$\zeta^{(s)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (5.3)$$

where we have chosen to label the rows and columns of the matrix as (135246). Similarly we can define a  $\zeta_7 = \sum_{ij} \zeta_{ij} Z_{ij7}$ . And we have after evaluating on the solution

$$\zeta_7^{(s)} = \frac{1}{\sqrt{18}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.4)$$

The variations of  $\zeta$  and  $\zeta_7$  are both obtained by the action of commutators as in the case of fuzzy  $S^4$ . Therefore an argument similar to the one above shows that  $U(3)$  Lie algebra matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

with  $A$  antisymmetric and  $B$  symmetric leave the solution invariant. As before the final step will rely on the tracelessness of the off-diagonal blocks of matrices of the form (4.6), where the blocks are now  $3 \times 3$  matrices.

So the stabilizer group is  $U(3)$ . By considering the action of the  $SO(7)$  symmetry of the equations satisfied by the variables  $Z_\mu$ ,  $Z_{\mu\nu}$ , and  $Z_{\mu\nu\lambda}$ , we find that a space  $SO(7)/U(3)$  is generated. This is a bundle over the sphere  $S^6$ . The fibre is  $SO(6)/U(3)$ .

$$\begin{aligned} SO(7)/U(3) &\longleftarrow SO(6)/U(3) \\ \downarrow & \\ S^6 &\equiv SO(7)/SO(6) \end{aligned} \tag{5.5}$$

### 5.2. Bundle structure at finite $n$

The fibre over the sphere  $S^6$  is, in the classical limit, the symmetric space  $SO(6)/U(3)$ . At finite  $n$  we have a fuzzy version of this space.

As in the discussion of  $\hat{\mathcal{A}}_n(S^4)$ , the vector subspace  $Sym(V_-^{\otimes n})$  can be identified with the points on the fibre. This space is spanned by  $|0\rangle$ , and four states of the form  $a_i^\dagger a_j^\dagger |0\rangle$ . The symmetric product is an irreducible representation of  $SO(6)$  with weights  $\vec{\lambda} = (\frac{n}{2}, \frac{n}{2}, \frac{n}{2})$ . This space has dimension ( see for example [19] )

$$D_6 = \frac{1}{6}(n+1)(n+2)(n+3). \tag{5.6}$$

The matrix algebra over this space is generated by the operators  $Z_{\mu\nu}$ . Multiplying these operators leads to operators of the form

$$\sum_{s_1, s_2, \dots, s_r} \rho_{s_1}(\Gamma) \rho_{s_2}(\Gamma) \cdots \rho_{s_r}(\Gamma). \tag{5.7}$$

This is somewhat schematic since we have not written out the indices on the  $\Gamma$  matrices. To be more precise we would write out the indices and contract with a traceless tensor which has the symmetries of the Young diagram with two rows of length  $r$ . Every pair of  $\Gamma$  in the same  $\rho$  factor is antisymmetrized. Such a representation has dimension

$$D(r) = \frac{1}{12}(2r+3)(r+2)^2(r+1)^2. \tag{5.8}$$

By adding up these dimensions in the range  $0 \leq r \leq n$  we get

$$\sum_{r=0}^n D(r) = D_6^2. \tag{5.9}$$

### 5.3. Relation to unitary bundles over the sphere

All these matrices are hermitian, so they form a basis for the Lie algebra of the unitary group with rank  $D_6 = \frac{1}{6}(n+1)(n+2)(n+3)$ . This indicates that the number of spherical 6-branes when we use  $\Phi_\mu \sim LZ_\mu$  is  $\frac{1}{6}(n+1)(n+2)(n+3)$ , giving a purely non-commutative geometric derivation of the 6-brane charge. As before, the  $Z_{\mu\nu}$  can be interpreted as field strengths for unitary bundles.

## 6. The fuzzy 8-sphere case

### 6.1. Solutions to equations and stabilizer group

The large  $N$  limit of the matrix algebra is generated by commuting variables,  $Z_\mu$ ,  $Z_{\mu\nu}$ ,  $Z_{\mu\nu\lambda}$  and  $Z_{\mu\nu\lambda\rho}$ . The indices run from 1 to 9 and any variable with more than one index is antisymmetric under exchange of any pair of indices.

By taking the expectation values of the appropriate matrices in a state one finds a particular solution :

$$\begin{aligned}
Z_9 &= 1, \\
\text{and } Z_i &= 0 \text{ for all other } i, \\
Z_{12} &= Z_{34} = Z_{56} = Z_{78} = \frac{1}{\sqrt{8}}, \\
\text{and } Z_{\mu\nu} &= 0 \text{ otherwise ,} \\
Z_{129} &= Z_{349} = Z_{569} = Z_{789} = \frac{1}{\sqrt{24}}, \\
\text{and } Z_{\mu\nu\rho} &= 0 \text{ otherwise ,} \\
Z_{1234} &= Z_{1256} = Z_{1278} = Z_{3456} = Z_{3478} = Z_{5678} = \frac{-1}{12} , \\
\text{and } Z_{\mu\nu\rho\lambda} &= 0 \text{ otherwise.}
\end{aligned} \tag{6.1}$$

By the statement that the variables with more than one index are zero otherwise, we mean that all components not related to the ones shown by the permutation symmetries are zero. For example  $Z_{219} = -Z_{912} = -\frac{1}{\sqrt{24}}$ , but  $Z_{139} = 0$ .

The normalizations are chosen such that

$$\begin{aligned}
\sum_{\mu} Z_{\mu}^2 &= 1, \\
\sum_{[\mu,\nu]} Z_{\mu\nu}^2 &= 1, \\
\sum_{[\mu,\nu,\lambda]} Z_{\mu\nu\lambda}^2 &= 1, \\
\sum_{[\mu,\nu,\lambda,\rho]} Z_{\mu\nu\lambda\rho}^2 &= 1.
\end{aligned} \tag{6.2}$$

The sums  $[\mu, \nu]$  etc. indicate that the indices run over 1 to 9 while respecting the condition that they are never equal. These variables are not all independent and obey constraints

like the one following from (3.11), and others that can be derived from the definitions in section 3.2 in the large  $n$  limit.

Now we consider the subgroup of  $SO(9)$  which leaves the solution invariant, i.e the stabilier group. Requiring that the  $Z_\mu$  values stay invariant clearly forces the stabilizer group to be a subgroup of  $SO(8)$ . Now requiring the  $Z_{\mu\nu}$  and  $Z_{\mu\nu\lambda}$  to be invariant picks out a  $U(4)$  subgroup of the  $SO(8)$ . The proof proceeds as for the case of the fuzzy  $S^6$ . This can be described by taking  $8 \times 8$  matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where  $B$  is symmetric and  $A$  is antisymmetric, and the rows and columns are labelled in the order (13572468). We still need to prove that the configuration of  $Z_{\mu\nu\lambda\rho}$  is invariant and this can be done directly using the following  $SO(9)$  transformations of the  $Z_{\mu\nu\lambda\rho}$

$$\begin{aligned} L_{\mu_1\mu_2}(Z_{\nu_1\nu_2\nu_3\nu_4}) &= (\delta_{\mu_2\nu_1}Z_{\mu_1\nu_2\nu_3\nu_4} - \delta_{\mu_2\nu_2}Z_{\mu_1\nu_1\nu_3\nu_4} + \delta_{\mu_2\nu_3}Z_{\mu_1\nu_1\nu_2\nu_4} - \delta_{\mu_2\nu_4}Z_{\mu_1\nu_1\nu_2\nu_3}) \\ &\quad - (\mu_1 \leftrightarrow \mu_2). \end{aligned} \tag{6.3}$$

The explicit proof is tedious, we just show some example steps. Take for example the combination  $L_{13} + L_{24}$  which is part of the  $U(3)$  subgroup. We have

$$\begin{aligned} L_{13}(Z_{3256}) &= Z_{1256}, \\ L_{24}(Z_{3256}) &= -Z_{3456}. \end{aligned} \tag{6.4}$$

Since  $Z_{1256} = Z_{3456}$  on the solution, this means that the variation by  $L_{13} + L_{24}$  on  $Z_{3256}$  is zero. Take another example, the variation of  $Z_{1356}$ . Now  $L_{13}(Z_{1356}) = L_{24}(Z_{1356})$  are both seen to be zero using (6.3) even before evaluating on the solution. Another example is the variation of  $Z_{3456}$ . The action of  $L_{13}$  on this is  $Z_{1456}$  which is zero on the solution, and the variation of  $L_{24}$  gives  $Z_{3156}$  which is also zero. Similarly we can check for each generator of the  $U(4)$  that the variation of any of the  $Z_{\mu\nu\lambda\rho}$  coordinates is zero. The steps are obvious and we will not spell out the complete proof.

Having established that the stabilizer group is  $U(4)$ , it follows that action of the  $SO(9)$  symmetry group generates a coset  $SO(9)/U(4)$ . This is a bundle over the 8-sphere.

$$\begin{aligned} SO(9)/U(4) &\longleftarrow SO(8)/U(4) \\ &\downarrow \\ S^8 &\equiv SO(9)/SO(8) \end{aligned} \tag{6.5}$$

### 6.2. Bundle structure of the coset at finite $n$

Viewed as a bundle over  $S^8$ , the coset  $SO(9)/U(4)$  has a fibre  $SO(8)/U(4)$ . At finite  $n$  we have, as a space of states which has the same expectation values for  $X_1, \dots, X_9$ , the space  $Sym(V_-^{\otimes n})$ , where  $V_-$  is spanned by states of the form  $|0\rangle$ ,  $a_i^\dagger a_j^\dagger |0\rangle$  and  $a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger |0\rangle$ . This symmetric tensor product space is an irreducible representation of  $SO(8)$  with weight  $\lambda = (\frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2})$  and dimension [19]

$$D_8 = \frac{1}{360}(n+5)(n+4)(n+3)^2(n+2)(n+2). \quad (6.6)$$

The operators  $Z_{\mu\nu}$  and  $Z_{\mu\nu\lambda\rho}$  generate the matrices acting on this space. We get operators of the form

$$\sum_{\vec{s}, \vec{t}} \rho_{s_1}(\Gamma\Gamma\Gamma\Gamma) \cdots \rho_{s_{p_1}}(\Gamma\Gamma\Gamma\Gamma) \rho_{t_1}(\Gamma\Gamma) \cdots \rho_{t_{p_2}}(\Gamma\Gamma). \quad (6.7)$$

After endowing operators of this form with  $SO(8)$  indices and contracting with traceless tensors of the appropriate symmetry, these correspond to irreducible representations associated with Young diagrams having row lengths  $(p_1 + p_2, p_1 + p_2, p_1, p_1)$ . This representation has dimension

$$D(p_1, p_2) = \frac{1}{4320}(2p_1 + 2p_2 + 5)(p_2 + 2)^2(p_2 + 2p_1 + 4)(p_2 + 3) \times \\ \times (p_2 + 3 + 2p_1)^2(p_2 + 1)(p_2 + 2p_1 + 2)(2p_1 + 1). \quad (6.8)$$

Adding up these dimensions, using Maple, we find

$$\sum_{p_1=0}^n \sum_{p_2=0}^{n-p_1} D(p_1, p_2) = D_8^2, \quad (6.9)$$

where  $D_8$  is given in (6.6).

These formulae allow us to read off the representations of  $SO(8)$  and their multiplicities (all 1) appearing as harmonics on the coset space  $SO(8)/U(4)$ .

### 6.3. Relation to unitary bundles

The generating operators  $Z_{\mu\nu}$  and  $Z_{\mu\nu\lambda\rho}$  are hermitian and their products which generate the full matrix algebra of dimension  $D_8^2$  are all hermitian. So they can be thought as Lie algebra elements for  $U(D)$ . This shows that a fuzzy 8-sphere construction based on these matrices in any matrix brane action involves  $D_8$  spherical 8-branes. As we discussed in the section on the fuzzy 4-sphere, we can read off the  $U(D)$  field strengths at the N-pole of the sphere.

## 7. Action for fluctuations and field theory on $SO(2k+1)/U(k)$

We will outline here some features of the action describing fluctuations around classical solutions of matrix brane world-volume theories, such as the solutions in [4], [3] or [2]. The details will depend on the particular application. We will be somewhat schematic, outlining generic features, in particular the appearance of the  $SO(2k+1)/U(k)$  coset as the base space of a field theory having  $U(1)$  gauge fields and an action compatible with the geometrical  $SO(2k+1)$  symmetry. For concreteness we will discuss the case of the four-sphere with  $SO(5)$  symmetry.

Consider either the application of [4] to brane intersections or the application in [2]. Take any scalar transverse to the 5 scalars that go into the fuzzy sphere. Call it  $\Phi$ . We expand it as  $\Phi = \sum_{R,s} a_{R,s} Y_{R,s}(Z)$ . The  $Y_{R,s}$ 's are operators associated with the state  $s$  in the representation  $R$  of  $SO(5)$ . They are normalized as

$$\int d\Omega \ Y_{R,s}(Z) Y_{R,s}(Z^\dagger) = 1. \quad (7.1)$$

To ensure correct normalization, operators of the form

$$\begin{aligned} \sum_{\mu_j^i} A[\mu_j^i] \sum_{\vec{s}} \rho_{s_1}(\Gamma^{\mu_1^1} \Gamma^{\mu_1^2}) \rho_{s_2}(\Gamma^{\mu_2^1} \Gamma^{\mu_2^2}) \cdots \rho_{s_{r_2}}(\Gamma^{\mu_{r_2}^1} \Gamma^{\mu_{r_2}^2}) \times \\ \times \rho_{s_{r_2+1}}(\Gamma^{\mu_{r_2+1}^1}) \cdots \rho_{s_{r_1}}(\Gamma^{\mu_{r_1}^1}), \end{aligned} \quad (7.2)$$

associated with Young diagrams having a first row of length  $r_1$  and a second row of length  $r_2$ , are multiplied by a factor  $N(n, r_1, r_2)$ , which behaves in the large  $n$  limit as  $n^{-r_1}$ .

From the kinetic term for  $\Phi$  we obtain

$$\frac{1}{g_s l_s} \int dt \ Tr \ (\partial_t \Phi)^2 = \frac{N}{g_s l_s} \int dt d\Omega \ (\partial_t \Phi)^2. \quad (7.3)$$

On the right hand side we have converted the trace into an integral, and we have recognized  $\Phi$  as a field living on the coset.

Other terms in the action give rise to spatial derivatives acting on the field  $\Phi$ . Indeed we have terms in the zero-brane action which are of the form  $\frac{1}{g_s l_s^5} \int dt Tr [\Phi_i, \Phi]^2$ . We expand  $\Phi_i = LZ_i + L^2 A_i$ , where  $L$  is the size of the sphere described by the matrix coordinates  $\Phi_i$  of the zero branes and  $A_i = \sum_{R,s} a_i^{R,s} Y_{R,s}(Z)$ . We chose a factor  $L^2$  in front of  $A_i$  to ensure that it has the dimensions of a gauge field. Since we identified  $X_i$  as derivatives section 3.3 we will get terms of the form

$$\frac{-N}{g_s l_s^5} \int dt d\Omega \ \left( \frac{L}{n} \left( Z_i \frac{\partial}{\partial Z_{ij}} + Z_{ij} \frac{\partial}{\partial Z_j} \right) \Phi \right)^2. \quad (7.4)$$

We also have terms of the form

$$\frac{L^4}{g_s l_s^5} \int dt d\Omega ([A_i, \Phi])^2. \quad (7.5)$$

This should be interpreted as a commutator of star products. More detailed formulae for the star product can be developed along the lines of [20] for example.

Similarly we have terms of the form  $\frac{1}{g_s l_s^5} \int dt Tr ([\Phi_i, \Phi_j])^2$ . These lead to expressions which include  $\int dt d\Omega (DA)^2$ ,  $\int dt d\Omega D^2$  and  $\int [A, A]^2$  terms, where the  $D$ 's are appropriate derivatives which can be read off from section 3. The term  $\int dt d\Omega D^2$  is just constant. The remaining terms are all terms we may expect from a theory containing gauge fields and scalars living on  $SO(5)/U(2)$  with an  $SO(5)$  symmetry and a non-commutative  $U(1)$  gauge symmetry.

The non-commutative  $U(1)$  gauge symmetry can be derived from the original symmetries of the Matrix Theory. We know that there is a unitary symmetry generated by hermitian matrices  $\Lambda$ . Since hermitian matrices give the spherical harmonics on the  $SO(5)/U(2)$  coset, we have symmetry variations of the form

$$\delta_\Lambda \Phi_i = [\Lambda, \Phi_i], \quad (7.6)$$

which translate into

$$\delta_\Lambda A_i = [X_i, \Lambda] + [A_i, \lambda], \quad (7.7)$$

where  $\Lambda$  is a function on the coset.

Note, however, that this  $U(1)$  gauge theory is different from ordinary gauge theories in that the gauge potential  $A_i$  is defined only for directions along the non-associative sphere, not for all directions on the coset. This is because there are only  $2k + 1$  matrix coordinates  $X_\mu$  in the original matrix model, which are Cartesian coordinates on a  $2k$ -sphere. The  $k(k - 1)$  new dimensions corresponding to fibres of the coset  $SO(2k + 1)/U(k)$  are generated by noncommutativity, which is dictated by the equations of motion for the matrix coordinates. This is reminiscent of the recent proposal of dynamically generated new dimensions [21].

## 8. Summary and outlook

We identified some higher dimensional geometries which are relevant to the matrix construction of fuzzy spheres of dimension greater than two. When we look at fuzzy 4-spheres, the relevant geometry is 6-dimensional. For 6-spheres it is 12 dimensional, and for 8-spheres it is 20 dimensional. These geometries of the form  $SO(2k+1)/U(k)$  are bundles over spheres  $S^{2k}$  with fibre  $SO(2k)/U(k)$ .

We outlined an approach to understand this geometry in terms of constructing 4-branes by locally putting together D0-branes, in the case of the fuzzy 4-sphere. It is shown that the fuzzy coset  $SO(5)/U(2)$  can be viewed locally as the product of fuzzy 2-sphere and non-associative 4-sphere. We generalized these to the 6-sphere and 8-sphere.

In developing the structure of the fuzzy  $SO(2k+1)/U(k)$  algebras as bundles over a non-associative sphere with a fuzzy  $SO(2k)/U(k)$  fibre, we were lead to an interpretation of the coordinates on the fuzzy fibre as field strengths on the base sphere for a unitary bundle whose rank could be read off from the geometry of the fuzzy fibre space. This gives a purely non-commutative geometric derivation of the brane charges involved. It also allows us to construct directly from the matrices the field strengths for the interpretation in terms of field theory on a sphere. For the case of the four sphere, this allowed us to reproduce the field strengths of the homogeneous instantons first discussed in this context in [4]. It would be interesting to develop this further to reproduce not just the field strength but also the connection. We find that the fuzzy spheres provide very peculiar examples of noncommutative space for which the noncommutativity of the non-Abelian gauge group is mixed with that of the base space.

Further study of the higher dimensional cases would also be interesting. The  $SO(2k+1)$  covariant Matrix Theory construction should allow the reconstruction of classical solutions for non-Abelian theories on  $S^{2k}$  in the large  $N$  limit.

The fluctuations of the solutions have a description in terms of non-Abelian theory of multiple branes on  $S^{2k}$ . Interestingly, there is also a description in terms of an Abelian theory on the *higher* dimensional coset space  $SO(2k+1)/U(k)$ . We outlined some elements of this description, noting the  $U(1)$  gauge symmetry and  $SO(2k+1)$  invariance. The detailed geometry of this Lagrangian remains to be understood. More generally, it will be interesting to further understand and interpret the appearance of these higher dimensional geometries in string theory.

While the fuzzy 4-sphere has been studied in some detail [2,4,22] in the context of Matrix constructions, involving a low-dimensional brane growing into a higher dimensional



brane, in contexts where several physical points of view are available, e.g the lower dimensional brane world-volume, the spherical brane worldvolume as well as supergravity, it will be very instructive to conduct analogous studies of more general spheres. Such generalizations should shed further light on the physics of the higher coset geometries  $SO(2k+1)/U(k)$ . Another potential class of applications of the various non-commutative geometries we discussed is purely field theoretic and attempts to find symmetry preserving finite approximations to base spaces of field theories. Some recent work on different examples of non-commutative cosets appears in [23][24], and metric aspects of the non-commutative geometry of spheres are studied in [25].

**Acknowledgements:** We wish to thank for pleasant discussions Steve Corley, Vladimir Fock, Zack Guralnik, Antal Jevicki, David Lowe and Djorje Minic. We also thank P. Etingof and I. Singer for email communications. The research of S.R. was supported in part by DOE grant DE-FG02/19ER40688-(Task A). The research of P.M.H. was supported in part by the National Science Council, R.O.C., the Center for Theoretical Physics at National Taiwan University, the National Center for Theoretical Sciences, and the CosPA project of the Ministry of Education, Taiwan, R.O.C.

## 9. Appendix : Jordan algebras and fuzzy spheres

We describe some further algebraic properties of  $\mathcal{A}_n(S^{2k})$ ,  $k \geq 2$ . For concreteness we describe the case  $\mathcal{A}_n(S^4)$ , but the arguments are general.

We first prove that the multiplication in  $\mathcal{A}_n(S^4)$  is commutative. Let us begin with some examples:

$$\begin{aligned}
X_\mu \circ X_\nu &= P\left(\sum_r \rho_r(\Gamma_\mu)\rho_s(\Gamma_\nu)\right) \\
&= P\left(\sum_r \rho_r(\Gamma_\mu\Gamma_\nu)\right) + P\left(\sum_{r \neq s} \rho_r(\Gamma_\mu)\rho_s(\Gamma_\nu)\right) \\
&= \sum_r \rho_r(\delta_{\mu\nu}) + \sum_{r \neq s} \rho_r(\Gamma_\mu)\rho_s(\Gamma_\nu) \\
&= n\delta_{\mu\nu} + \sum_{r \neq s} \rho_r(\Gamma_\mu)\rho_s(\Gamma_\nu) \\
&= \frac{(n^2 + 4n)}{5}\delta_{\mu\nu} + X_{(\mu\nu)}.
\end{aligned} \tag{9.1}$$

In the third line, we have used the fact that the antisymmetric part of the first term belongs to a representation with row lengths  $\vec{r} = (1, 1)$  which is projected out by  $P$ . This leaves

only the symmetric part which is proportional to  $\delta_{\mu\nu}$ . In the last line, we have introduced a symmetric traceless tensor of rank 2

$$X_{(\mu\nu)} = \sum_{r \neq s} \rho_r(\Gamma_\mu) \rho_s(\Gamma_\nu) - \delta_{\mu\nu} \frac{n(n-1)}{5}. \quad (9.2)$$

It is clear that the product in (9.1) is commutative. In general we have a product of the form

$$\begin{aligned} X_{(\mu_1 \mu_2 \dots \mu_k)} X_{(\nu_1 \nu_2 \dots \nu_l)} &= a_1 X_{(\mu_1 \mu_2 \dots \mu_k \nu_1 \nu_2 \dots \nu_l)} \\ &+ a_2 (\delta_{\mu_1 \nu_1} X_{(\mu_2 \dots \mu_k \nu_2 \dots \nu_l)} + \delta_{\mu_2 \nu_2} X_{(\mu_1 \mu_3 \dots \mu_k \nu_1 \nu_3 \dots \nu_l)} + \dots) + \dots, \end{aligned} \quad (9.3)$$

where  $a_1, a_2$  etc. are constants that can be worked out. The remaining terms multiplied by  $a_2$  involve different contractions of one  $\mu$  index with one  $\nu$  index. The “...” denote terms with more  $\delta$ 's which contract some set of  $\mu$  indices with some set of  $\nu$  indices multiplied by symmetric traceless tensors with lower rank. All the terms appearing on the right hand side of (9.3) are symmetric under the operation which exchanges all the  $\mu$  with the  $\nu$  indices. This shows that the product is commutative for an arbitrary pair of elements in  $\mathcal{A}_n(S^4)$ .

It is important to note that while the product  $A \circ B$  is commutative, it does not contain the most general element of  $\hat{\mathcal{A}}_n(S^4)$  appearing symmetrically in the matrix product.

$$A \circ B \neq \frac{1}{2}(AB + BA). \quad (9.4)$$

We recall that the product  $A * B = \frac{1}{2}(AB + BA)$  actually make  $\hat{\mathcal{A}}_n(S^4)$  a Jordan Algebra, which requires that the following identity be satisfied ( see for example [26])

$$((A * A) * B) * A = (A * A) * (B * A). \quad (9.5)$$

The product  $A \circ B$  has an interesting relation to  $A * B$ :

$$A \circ B = A * B - Q(AB). \quad (9.6)$$

$Q$  is a projector which acts as 1 on all elements of  $\hat{\mathcal{A}}_n(S^4)$  which transform in representations of  $SO(5)$  corresponding to Young diagrams with rows  $\vec{r} = (2L + M, 2L)$  for  $L \geq 1$  and  $M$  is an arbitrary integer such that  $2L + M \leq n$ . Equivalently, these Young diagrams have an even ( non-zero ) number of columns of length 2.  $Q$  is zero on all other representations. Representations which are picked out by  $Q$ , while they involve some antisymmetrizations

and are projected out by  $P$ , have the property that they can appear symmetrically when  $A$  and  $B$ , belonging to the vector space  $\mathcal{A}_n(S^4)$ , are multiplied as matrices.

We observe that  $Q(X_\mu X_\nu) = 0$ . Representations with an even number of columns of length two do not appear in the matrix product  $X_\mu X_\nu$ , which means that the fuzzy sphere product coincides with the symmetrized product and the identity in (9.5) is satisfied if  $A$  and  $B$  involve only  $X_\mu, X_\nu$ . When we multiply more general elements in  $\mathcal{A}_n(S^4)$  such as  $X_{(\mu\nu)} \circ X_{(\alpha\beta)}$  we can get the representations picked out by  $Q$ .

Representations which have  $\vec{r} = (r_1, r_2)$  with  $r_1, r_2 \ll n$ , give harmonics which must be normalized by factors  $\mathcal{O}(1/n^{r_1})$ , since they involve operators acting on  $r_1$  different factors of the  $n$ -fold tensor product. Keeping track of such normalizations

$$\frac{X_{[k]}}{n^k} \frac{X_{[l]}}{n^l} \sim \frac{1}{n^2} \frac{X_{[k+l-2, 2]}}{n^{k+l-2}} + \dots, \quad (9.7)$$

we have written down the simplest operator picked out by  $Q$  which appears when symmetric traceless tensor operators of rank  $k$  and  $l$  are multiplied ( the subscript on the  $X$  describe the row lengths of the irreducible representation they transform in ). If we only keep terms at order  $1/n$  these can be neglected from the matrix product, so  $A \circ B = A * B$ , which means that the Jordan identity is satisfied. Recall that the simplest representations which cause non-associativity of the  $A \circ B$  product are the representations with a second row of length 1 which can appear in the matrix product. This is a  $1/n$  effect.

To summarize, if we neglect all terms in the structure constants of the normalized operators which are  $\mathcal{O}(1/n)$  and higher, we have a commutative and associative multiplication as expected for an algebra approaching the classical algebra of functions on the  $S^4$ . If we keep  $\mathcal{O}(1/n)$  and neglect terms of  $\mathcal{O}(\frac{1}{n^2})$  and higher, we have commutative, non-associative algebra which is an infinite dimensional Jordan algebra. If we keep higher order terms we have a commutative and non-associative algebra which is not Jordan. And this is the structure at generic finite  $n$ .

## References

- [1] H. Grosse, C. Klimcik and P. Presnajder, “On finite 4-D Quantum Field Theory in non-commutative geometry,” [hep-th/9602115], Commun.Math.Phys.180:429-438,1996.
- [2] J. Castellino, S. Lee, W. Taylor, “Longitudinal 5-branes as 4-spheres in matrix theory,” [hep-th/9712105], Nucl.Phys. B526 (1998) 334-350.
- [3] P. M. Ho, “Fuzzy sphere from matrix model,” [hep-th/0010165], JHEP **0012**, 015 (2000).

- [4] N. Constable, R. Myers and O. Tafjord, “Non-abelian brane intersections,” [hep-th/0102080], JHEP 0106:023,2001.
- [5] Z. Guralnik, S. Ramgoolam, “On the polarization of unstable D0-branes into non-commutative odd spheres,” [hep-th/0101001], JHEP 0102 (2001) 032.
- [6] A. Jevicki and S. Ramgoolam, “Noncommutative gravity from the ADS/CFT correspondence,” [hep-th/9902059], JHEP **9904** (1999) 032.
- [7] P.M.Ho, S.Ramgoolam and R.Tatar, “Quantum space-times and finite N effects in 4-D superyang-mills theories,” [hep-th/9907145], Nucl.Phys.**B573** (2000) 364,
- [8] M. Berkooz and H. Verlinde, “Matrix theory, AdS/CFT and Higgs-Coulomb equivalence,” [hep-th/9907100], JHEP 9911 (1999) 037.
- [9] P. M. Ho and M. Li, “Fuzzy spheres in AdS/CFT correspondence and holography from noncommutativity,” Nucl. Phys. B **596**, 259 (2001) [hep-th/0004072]. P.-M. Ho and M. Li, “Large N expansion from fuzzy AdS(2),” Nucl. Phys. B **590**, 198 (2000) [hep-th/0005268].
- [10] S. Ramgoolam, “On spherical harmonics for fuzzy spheres in diverse dimensions,” Nucl. Phys. B **610**, 461 (2001) [hep-th/0105006].
- [11] B. de Wit, J. Hoppe, H. Nicolai, “On the quantum mechanics of supermembranes,” Nucl.Phys.B305:545,1988.
- [12] R. C. Myers, “Dielectric-branes”,JHEP 9912 (1999) 022 [hep-th/9910053].
- [13] M. Kramer, “Some remarks suggesting an interesting theory of harmonic functions on  $SU(2n+1)/Sp(n)$  and  $SO(2n+1)/U(n)$ ,” Arch. Math. 33 (1979/80), 76-79.
- [14] I.M.Gelfand, A.V. Zelevinskii, “Models of representations of classical groups and their hidden symmetries,” Funct. An. Priloz. vol. 18, 1984.
- [15] A. Connes, “Non-commutative geometry,” Academic Press, 1994.
- [16] O. Ganor, S. Ramgoolam, W. Taylor, “Branes, fluxes and duality in M(atrrix)-theory,” [hep-th/9611202], Nucl.Phys.B492:191-204,1997
- [17] T. Banks, N. Seiberg, S. Shenker, “Branes form Matrices,” [hep-th/9612157], Nucl.Phys. B490:91-106,1997
- [18] H. C. Wang, Nagoya. Math. J. **13**, 1 (1958).
- [19] W. Fulton and G. Harris, “Representation theory,” Springer Verlag 1991.
- [20] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, “Noncommutative gauge theory on fuzzy sphere from matrix model,” Nucl. Phys. B **604**, 121 (2001) [hep-th/0101102].
- [21] N. Arkani-Hamed, A. G. Cohen and H. Georgi, “(De)constructing dimensions,” [hep-th/0104005], Phys. Rev. Lett. **86**, 4757 (2001).
- [22] P.J.Silva, “Matrix string theory and the Myers effect,” [hep-th/0111121].
- [23] A.P. Balachandran, B. P. Dolan, J.H. Lee, X. Martin, D. O’Connor “Fuzzy complex projective spaces and their star products,” [hep-th/0107099]

- [24] S. P. Trivedi, S. Vaidya, “Fuzzy cosets and their gravity duals,” [hep-th/0007011], JHEP 0009:041,2000
- [25] M.A.Rieffel “Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance,” [math.OA/0108005].
- [26] F. Gursey and C.H. Tze, “On the role of division, Jordan and related algebras in particle physics,” World Scientific 1996.